Class 9, given on Jan 22, 2010, for Math 13, Winter 2010

Having spent a considerable amount of time studying how to evaluate all sorts of different kinds of double integrals, we now briefly list some typical applications of double integration to physics and engineering. As a matter of fact, calculus was invented to allow scientists (such as Newton) the ability to calculate quantities from physics they were interested in.

Suppose we are given a *lamina*, which is a two-dimensional distribution of mass in some sort of shape. The shape will typically be specified by a domain D in the xy plane, and the distribution of mass might be specified by a density function $\rho(x, y)$ defined on D. Recall that the (average) density of a two-dimensional object with mass m and area A is given by m/A. The density function $\rho(x, y)$ can be interpreted as the limit of the densities of smaller and smaller squares around the point (x, y). (We ignore the issue of what units we use. As long as we are consistent with the choice of units for each of the quantities we are interested in, all the formulas below will work as is.)

Of course, in real life, objects are three-dimensional. Nevertheless, the simplification to two dimensions is instructive, because many of the calculations we perform will carry over to the three-dimensional case with little modification, yet be easier to do. Also, in some physical applications, it is sufficient to approximate a thin three-dimensional object as simply a two-dimensional object.

So suppose we have a lamina in the shape of a domain D with density function $\rho(x, y)$. A common question is to calculate the mass of the lamina. Because mass is equal to density times area, we can approximate the mass of a lamina with a Riemann sum

$$\sum \rho(x^*, y^*)\Delta x\,\Delta y$$

where we cut up the lamina into lots of small rectangles and then approximate the mass of each rectangle by multiplying the density at some point ($\rho(x^*, y^*)$ in the rectangle by the area of the rectangle ($\Delta x\,\Delta y$). As we take finer and finer approximations, the limit of these Riemann sums approaches the double integral

$$\iint_D \rho(x, y)\,dA.$$

This is the mass of the lamina, and we will often call it m.

Example. Suppose we have a lamina which is a triangle with vertices $(0, 0), (0, 1), (1, 1)$, and density function $\rho(x, y) = y$. What is the mass of the lamina?

The formula for mass tells us that we should evaluate the double integral

$$\iint_D y\,dA.$$

The region D is defined by inequalities $0 \le x \le 1, 0 \le y \le 1 - x$, so this double integral equals the iterated integral

$$\int_0^1 \int_0^{1-x} y\,dy\,dx = \int_0^1 \frac{y^2}{2}\Big|_{y=0}^{y=1-x} dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{-(1-x)^3}{6}\Big|_0^1 = \frac{1}{6}.$$

One of the interesting facts about physics is that different physical phenomena are governed by similar, or even the same, mathematical rules. For example, the forces of gravity and electricity act in very similar fashions – they both obey inverse square force laws. The discussion above could have worked just as well for a lamina of shape D with *charge distribution* $\rho(x, y)$, and then the double integral for mass above would instead calculate the total charge on the lamina. (One difference between gravity and electricity is that when dealing with mass, $\rho(x, y) \geq 0$, but when dealing with electric charge, ρ can be negative.) So even though we are calculating a different physical quantity, the mathematical calculations are identical.

Another quantity of interest associated to a lamina is its *center of mass*. This is the point at which the lamina can be balanced on a point, for example. Another interpretation of the center of mass of an object is that, when considering the force of gravity the object exerts on other objects, that force acts as if the entire mass of the lamina were concentrated at the center of mass. This was one of the problems calculus was initially created to solve – to simplify the calculation of gravitational forces exerted by various objects.

To help us calculate the center of mass of a lamina, we define the *moment about the x-axis* (respectively, moment about the y-axis) – this is not the moment of inertia! – to be the value of the integral

$$M_x = \iint\limits_D y\rho(x, y)\, dA, M_y = \iint\limits_D x\rho(x, y)\, dA.$$

Then the center of mass of the lamina has coordinates $\overline{x}, \overline{y}$ given by

$$\overline{x} = \frac{M_y}{m}, \overline{y} = \frac{M_x}{m}$$

Example. Consider a triangle with vertices at $(0, 0), (3, 0)$, and $(0, 5)$, of uniform density (say $\rho(x, y) = 1$). Where is the center of mass of this triangle located?

Let D be the triangle; it is determined by inequalities $0 \leq x \leq 3, 0 \leq y \leq 5 - 5x/3$. The mass of the triangle is easy to calculate in this situation; the triangle has area $3(5)/2 = 15/2 = 7.5$, so the mass is also $m = 7.5$.

The moment of the triangle with respect to the x axis is given by

$$\int_0^3 \int_0^{5-5x/3} y\, dy\, dx = \int_0^3 \frac{y^2}{2}\bigg|_{y=0}^{y=5-5x/3} dy = \int_0^3 \frac{(5 - 5x/3)^2}{2}\, dy.$$

This is equal to

$$\frac{(5 - 5x/3)^3}{6} \cdot \frac{-3}{5}\bigg|_0^3 = \frac{5^3}{6} \cdot 35 = \frac{25}{2}.$$

Therefore the y coordinate of the center of mass is

$$\overline{y} = \frac{25}{2} \cdot \frac{2}{15} = \frac{5}{3}.$$

Similarly, the moment of the triangle with respect to the y axis is given by

$$\int_0^3 \int_0^{5-5x/3} x\, dy\, dx = \int_0^3 x(5 - 5x/3)\, dx = \frac{5x^2}{2} - \frac{5x^3}{9}\bigg|_0^3 = \frac{45}{2} - \frac{15}{=}\frac{15}{2}.$$

Therefore, the x coordinate of the center of mass is given by

$$\overline{x} = \frac{15}{2} \cdot \frac{2}{15} = 1.$$

Why such a funny triangle? We can 'experimentally' verify our calculations by taking a 3×5 index card, cutting it along a diagonal, and then try to balance the resulting triangle at the calculated center of mass. Sure enough, we find that mathematical theory agrees with real life!

Example. Let D be the half-annulus $9 \leq x^2 + y^2 \leq 16, y \geq 0$. Suppose we have a lamina whose shape is D and has uniform density. Find the center of mass.

The area of D is given by $16\pi - 9\pi/2 = 7\pi/2$, since we can take half the area of the difference of a circle of radius 4 and a circle of radius 3. (Of course, you could also evaluate a double integral, but the resulting calculation is longer than using ordinary geometry.) Since the region D involves circles, we will switch to polar coordinates. The polar inequalities which define D are $3 \leq r \leq 4, 0 \leq \theta \leq \pi$.

Symmetry strongly suggests that $\overline{x} = 0$, but we can calculate this quickly as well. The moment about the y-axis is given by

$$\iint_D y\, dA = \int_0^\pi \int_3^4 r^2 \cos\theta\, dr\, d\theta.$$

If you evaluate the inner integral, you will get a constant times $\cos\theta$. Notice that the integral of $\cos\theta$ over $[0,\pi]$ is equal to 0. Therefore, $\overline{x} = 0$ as symmetry suggests.

The moment about the x-axis is given by

$$\iint_D x\, dA = \int_0^\pi \int_3^4 r^2 \sin\theta\, dr\, d\theta = \int_0^\pi \frac{64}{3} - \frac{27}{3} \sin\theta\, d\theta = \int_0^\pi \frac{37}{3} \sin\theta\, d\theta = \frac{37}{3} \cdot 2 = \frac{74}{3}.$$

Therefore, the y coordinate of the center of mass is

$$\overline{y} = \frac{74}{3} \cdot \frac{2}{7\pi} = \frac{37}{7\pi} \approx 1.682.$$

Notice that the center of mass does not lie in D, and is actually below D. This principle is well-known to high jumpers, who arch their backs in a semi-circular fashion as they clear the bar. This allows them to have every part of their body above the bar while keeping the center of mass below the bar.

Another quantity of interest when studying rotational motion (torque, etc.) is the moment of inertia. For a point mass, the moment of inertia about a given axis ℓ of a point with mass m of distance r from ℓ is defined to be mr^2. Therefore, we define the moment of inertia of a lamina which occupies D and has density function $\rho(x,y)$ about the x-axis to be

$$I_x = \iint_D y^2 \rho(x,y)\, dA.$$

Similarly, the moment of inertia about the y-axis is

$$I_y = \iint_D x^2 \rho(x,y)\, dA.$$

The moment of inertia about the origin is given by

$$\iint_D (x^2 + y^2)\rho(x,y)\, dA = I_x + I_y.$$

This can also be thought of as the moment of inertia about the z-axis, if we think of the lamina D as lying in the xy plane in $\mathbb{R}^3$.

Example. Wikipedia (or any introductory physics textbook) says that the moment of inertia of a disc about the x axis (or y axis, by symmetry) with radius R, centered at the origin, of uniform density, is given by $mR^2/4$. Show that this formula is correct.

Let $\rho(x, y) = 1$. Then the mass of the disc is given by $m = \pi R^2$. The moment of inertia about the x-axis is given by

$$\iint_D y^2 \, dA.$$

Since D is circular, we use polar coordinates to evaluate this double integral. D is given by polar inequalities $0 \leq r \leq R, 0 \leq \theta \leq 2\pi$. Therefore, the iterated integral we want to calculate is

$$\int_0^{2\pi} \int_0^R (r \cos \theta)^2 r \, dr \, d\theta.$$

Evaluating the inner integral is easy: we get

$$\int_0^{2\pi} \frac{R^4}{4} \cos^2 \theta \, d\theta.$$

The $R^4/4$ is a constant. We now need to integrate $\cos^2 \theta$. This requires the use of the trigonometric identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}.$$

If we use this, we obtain

$$\frac{R^4}{4} = \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{R^4}{4} \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \Big|_0^{2\pi} = \frac{R^4}{4}(\pi + 0).$$

Since the mass $m = \pi R^2$, the moment about the x-axis can be rewritten as

$$\frac{mR^2}{4},$$

as desired.

As a matter of fact, you can now verify all sorts of formulas that you were given in physics class for centers of mass and moments of inertia using integration, and also calculate these quantities for a wider class of two-dimensional objects. One great feature of the mathematics behind the physics is that the mathematics is, by and large, the same, regardless of what quantities you are interested in calculating. For example, to compute the mass, center of mass, and moments of inertia of a lamina, you compute various integrals. Although the integrals are different, the techniques for calculating them are very similar, and you do not need to learn new mathematical ideas.

In a few weeks we will quickly see how to calculate these quantities for three-dimensional objects, or for that matter, n-dimensional objects (although $n = 3$ is by far the most common case, since it corresponds with the physical reality we observe) using triple or n-fold integrals.